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A conjecture of Welsh revisited

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ABSTRACT

Welsh conjectured that for any simple regular connected matroid M, if each cocircuit has at least $\frac{1}{2}(r(M)+1)$ elements, then there is a circuit of size $r(M)+1$. This conjecture was proven by Hochstättler and Jackson in 1997. In this paper, we give a shorter proof of this conjecture based solely on matroid-theoretical arguments. Let M be a simple, connected, regular matroid and let $C \in \mathcal{C}(M)$, where $|C| \leq \min\{r(M), 2d-1\}$. We show that if $|C^*| \geq d \geq 2$, $\forall C^* \in \mathcal{C}^*(M)$ where $C \cap C^* = \emptyset$, then there is a circuit D such that $D \triangle C$ is a circuit where $|D \triangle C| > |C|$.

1. Introduction

In [1], Dirac introduced a fundamental theorem establishing sufficient conditions for the existence of Hamilton cycles in graphs.

Theorem 1.1 (*Dirac*)**.** *Let G be a simple graph of order $n \geq 3$. If $d_G(v) \geq \frac{n}{2}$ for all vertices $v \in V(G)$, then G has a Hamilton cycle.*

Motivated by Dirac's theorem, Welsh (see [4], Problem 14.4.1) conjectured that a more general result holds for matroids:

Conjecture 1.2 (*Welsh*)**.** *If M is a simple regular connected matroid and every cocircuit has at least $\frac{1}{2}(r(M)+1)$ elements, then M has a circuit of size $r(M)+1$.*

In [2], Hochstättler and Jackson verified Welsh's conjecture. They also proved the following theorem:

Theorem 1.3 (*Hochstättler, Jackson*)**.** *Let M be a simple connected binary matroid such that every cocircuit has size at least $d \geq 3$. If M has no F_7-minor, $M \neq F_7^*$, and $d \notin \{5,6,7,8\}$, then M has a circuit of size at least $\min\{r(M)+1, 2d\}$.*

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In their proof of the above theorem, they use Seymour's decomposition theorem for regular matroids (see [4, Theorem 13.2.4] or [5]) to reduce the problem to the case of graphic and cographic matroids which they treat separately. In this paper, we give a shorter, simpler proof of Welsh's conjecture which avoids any complicated decomposition results. We shall prove the following theorem, from which Welsh's conjecture is a consequence:

Theorem 1.4. *Let M be a simple connected regular matroid and let $C \in \mathcal{C}(M)$, where $|C| \leq \min\{r(M), 2d-1\}$. Suppose $|C^*| \geq d \geq 2$, $\forall C^* \in \mathcal{C}^*(M)$, $C \cap C^* = \emptyset$. Then there is a circuit D such that $C \triangle D$ is a circuit where $|C \triangle D| > |C|$.*

In this paper, we actually prove a slightly stronger version of this theorem (see Theorem 3.2). We remark that the condition $d \notin \{5, 6, 7, 8\}$ in Theorem 1.3 is not needed in the above theorem. To remove this condition, it was conjectured in [2, Conjecture 1] that every essentially 4-connected graph with n vertices, m edges, minimum degree at least three, and girth $d \in \{5, 6, 7, 8\}$ where $2d \geq m - n + 2$ has a cocircuit of size at least $2d$. Theorem 1.4 indirectly verifies this conjecture.

2. Crossing sets and F_7-minors

We shall first define some concepts and notation for sets. Let A and B be subsets of a set X. If X is the disjoint union of A and B, then we shall write $X = A \overset{\circ}{\cup} B$. We say that A and B **cross** in X if the sets $A \cap B$, $(X \backslash A) \cap B$, $A \cap (X \backslash B)$, and $(X \backslash A) \cap (X \backslash B)$ are all nonempty subsets. In shorthand, we write $A \pitchfork B$, the set X being implicitly known in most cases. If A does not cross B, then we write $A \not\pitchfork B$. If either $A \subseteq B$ or $B \subseteq A$, then we shall write $A \diamond B$. If this is not the case, we write $A \not\diamond B$. Furthermore, if $A \cap B = \emptyset$ or $A \cup B = X$, then we shall write $A \parallel B$. If this is not the case, we write $A \nparallel B$. It is seen that $A \diamond B$ (respectively, $A \parallel B$) if and only if $(X \backslash A) \diamond (X \backslash B)$ (respectively, $(X \backslash A) \parallel (X \backslash B)$). It is also straightforward to see that $A \not\pitchfork B$ if and only if $A \diamond B$ or $A \parallel B$. A collection of sets $\mathcal{S}$ is non-crossing if each pair of sets in $\mathcal{S}$ is non-crossing. A **k-circuit** is defined to be a circuit having k elements. A 2-circuit will also be called a **digon** and a 3-circuit is also called a **triangle**. In the proof of the main result (Theorem 3.2), the following lemmas will be instrumental.

Lemma 2.1. *Let M be a binary matroid. Suppose C_1, C_2, C_3 are triangles intersecting in a common element e, and $M \mid C_1 \cup C_2 \cup C_3$ is simple. If there exists a 3-circuit C_4 where $e \notin C_4$ and C_4 intersects each of C_1, C_2, C_3, then $M \mid C_1 \cup C_2 \cup C_3 \simeq F_7$.*

Proof. This follows from three applications of the fact that if 123, 345, and 561 are circuits in a binary matroid, then 246 is also a circuit. □

Let M be a simple connected binary matroid and let C be a circuit. Let $C_1, C_2, C_3 \in \mathcal{C}(M)$ where $e \in C_1 \cap C_2 \cap C_3$ and let f_i, $i = 1, 2, 3$ be distinct elements where $C_i \backslash (C \cup \{e\}) = \{f_i\}$, $i = 1, 2, 3$. If

(i) $C_i \triangle C$, $i = 1, 2, 3$ are circuits of M
and
(ii) there exist distinct elements $g_i \in C_i \cap C$, $i = 1, 2, 3$ where $g_i \in C_j$ if and only if $i = j$,

then we write $(g_1, g_2, g_3) \underset{C,e}{\rightarrow} (C_1, C_2, C_3)$.

Lemma 2.2. *If $(g_1, g_2, g_3) \underset{C,e}{\rightarrow} (C_1, C_2, C_3)$, then M has an F_7-minor.*

Proof. Let $N = M \mid C \cup C_1 \cup C_2 \cup C_3$, and let $N' = N/(C \backslash \{e, g_1, g_2, g_3\})$. Then $\{e, f_1, g_1\}$, $\{e, f_2, g_2\}$, $\{e, f_3, g_3\}$ are seen to be circuits of N' and N' is simple. If $e \in C$, then $\{e, g_1, g_2, g_3\}$ is also a circuit of N' and $\{f_1, f_2, f_3\} = \{e, f_1, g_1\} \triangle \{e, f_2, g_2\} \triangle \{e, f_3, g_3\} \triangle \{e, g_1, g_2, g_3\}$ is also a circuit (because N' is simple) which intersects the circuits $\{e, f_1, g_1\}$, $\{e, f_2, g_2\}$, $\{e, f_3, g_3\}$ containing e. It follows from Lemma 2.1 that $N' \simeq F_7$. If $e \notin C$, then $\{g_1, g_2, g_3\}$ is a circuit of N' which intersects the triangles $\{e, f_1, g_1\}$, $\{e, f_2, g_2\}$, and $\{e, f_3, g_3\}$. Again, Lemma 2.1 implies that $N' \simeq F_7$. □

3. Augmenting circuits

Let M be a connected binary matroid with ground set E. Let $C \in \mathcal{C}(M)$, and let $X \subseteq E \backslash C$. A circuit D is called an **(X, C)-circuit** if $D \cap X \neq \emptyset$, and $C \cap D \neq \emptyset$. We say that a circuit D is **C-minimal** if C, D, and $C \triangle D$ are exactly the circuits contained in $C \cup D$. We observe that if D is a C-minimal (X, C)-circuit, then $C \triangle D$ is a also a C-minimal (X, C)-circuit. We say that a circuit D is **C-augmenting** if it is a C-minimal circuit for which $|C \triangle D| > |C|$. In connection with C-minimal circuits, we have the following lemma from [3]:

Lemma 3.1. *Let M be a connected binary matroid having no F_7^*-minor. Let $C \in \mathcal{C}(M)$ and $X \subseteq E(M) \backslash C$. Let D be a C-minimal (X, C)-circuit for which $|D \cap C|$ is minimum. Let $D_0 = D \backslash C$, $D_1 = D \cap C$, and $D_2 = C \backslash D$. Then D_0 and D_1 are circuits of M/D_2 belonging to different components of M/D_2.*

In the sections to follow, we shall prove the next theorem which is a slight strengthening of Theorem 1.4.

Theorem 3.2. *Let M be a simple connected regular matroid and let $C \in \mathcal{C}(M)$, where $|C| \leq \min\{r(M), 2d - 1\}$. Suppose $|C^*| \geq d \geq 2$, for all $C^* \in \mathcal{C}^*(M)$, where $C \cap C^* = \emptyset$. Then there exists a C-augmenting circuit.*

4. Properties of a minimum counterexample

We shall prove Theorem 3.2 by using the minimum counterexample approach. We suppose that the theorem is false, and we let M be a simple connected regular matroid which is a counterexample for which $|E(M)|$ is minimum. Since the theorem is seen to hold when $|E(M)| \leq 5$, we have that $|E(M)| \geq 6$. Let C be a circuit of M where $|C| \leq \min\{r(M), 2d - 1\}$, and $|C^*| \geq d$ for all $C^* \in \mathcal{C}^*(M)$ where $C \cap C^* = \emptyset$. We shall assume that there are no C-augmenting circuits. In this section, we shall show that M has certain properties.

(4.1). *M/e is connected $\forall e \in C$.*

Proof. Let $e \in C$. If M/e is not connected, then there is a C-minimal circuit D where $C \cap D = \{e\}$. In this case, $|C \triangle D| = |D| + |C| - 2 > |C|$, and D is C-augmenting, contradicting our assumption. □

(4.2). *There exists a C-minimal circuit D for which $|C \triangle D| = |C|$.*

Proof. Let $e \in C$ and $C' = C \backslash \{e\}$. By (4.1), M/e is connected. Suppose M/e is not simple, and let $\{f, g\}$ be a digon of M/e. Then $\{e, f, g\}$ is a triangle of M. If $f, g \in C^*$ for some $C^* \in \mathcal{C}^*(M)$, where $C \cap C^* = \emptyset$. Then $\{e, f, g\}$ is a C-minimal circuit where $|\{e, f, g\} \triangle C| = |C| + 1$. This implies that $\{e, f, g\}$ is C-augmenting, contradicting our assumptions. Hence there is no cocircuit C^* disjoint from C where $f, g \in C^*$. More generally, there is no cocircuit C^* disjoint from C where C^* contains a digon of M/e. In particular, if $\widetilde{M/e}$ is the simplification of M/e, then every cocircuit $C^* \in \mathcal{C}^*(\widetilde{M/e})$ where $C \cap C^* = \emptyset$, is also a cocircuit of M/e, and hence also of M. Thus for all $C^* \in \mathcal{C}^*(\widetilde{M/e})$ where $C \cap C^* = \emptyset$, it holds that $|C^*| \geq d$. Since $|E(\widetilde{M/e})| < |E(M)|$, there is a C'-augmenting circuit D' in $\widetilde{M/e}$. Suppose D' is also a circuit of M. Then D' is seen to be a C-minimal circuit. Moreover, $|C \triangle D'| = |C' \triangle D'| + 1 > |C'| + 1 = |C|$. Thus D' is C-augmenting, contradicting our assumptions. Consequently, D' is not a circuit of M and hence $D = D' \cup \{e\}$ is a circuit. It is seen that D is a C-minimal circuit. We have $|C \triangle D| = |C' \triangle D'| > |C'| = |C| - 1$. Thus $|C \triangle D| \geq |C|$, and given that D is not augmenting, it follows that $|C \triangle D| = |C|$. □

Let D be a C-minimal circuit for which $|C \triangle D| = |C|$. In addition, choose D so that $|C \cap D|$ is minimum among all such circuits. Let $D_0 = D \backslash C$, $D_1 = D \cap C$, $D_2 = C \backslash D$, $M' = M/D_2$. It should be noted that $|D_1| \geq 2$; if $|D_1| = 1$, then $|C \triangle D| = |C| + |D| - 2 \geq |C| + 1$ (since $|D| \geq 3$).

(4.3). $|D_0| = |D_1|$. *Moreover, if $|C| = r(M)$, then $|D_0| = |D_1| = 2$.*

Proof. We have $|D_0| + |D_2| = |C \triangle D| = |C| = |D_1| + |D_2|$, and hence $|D_0| = |D_1|$. If $|C| = r(M)$, then for any two elements $e, f \in D_0$ it holds that $\{e, f\} \cup C$ contains a circuit containing e and f. Thus it must hold that $D_0 = \{e, f\}$, and $|D_0| = |D_1| = 2$. □

We observe that D_0 and D_1 are circuits of M', each having the same cardinality.

(4.4). *D_0 and D_1 belong to different components of M'.*

Proof. By straightforward modifications to the proof of Lemma 3.1 in [3, Lemma 2.2] one can prove the assertion. We omit the details. □

Let K' be the component of M' containing D_1. Let $N = M \backslash K'$ and $C_N = D_0 \cup D_2$. Observe that by (4.3), $|C_N| = |C|$.

(4.5). *C_N is a spanning circuit of N.*

Proof. By contradiction. Suppose C_N is not a spanning circuit of N and $|C_N| \leq r(N)$. Then there exists $C_N^* \in \mathcal{C}^*(N)$ where $C_N^* \cap C_N = \emptyset$. Suppose $C_N^* \notin \mathcal{C}^*(M)$. Then $C^* = C_N^* \cup A$ is a cocircuit of M for some $\emptyset \neq A \subset K'$. We have $C^* \cap D_2 = \emptyset$, and hence it holds that $C^* \in \mathcal{C}^*(M')$. Since $A = C^* \cap K' \neq \emptyset$, and K' is a component of M', it must hold that $C^* \subset K'$, which is clearly impossible since $C_N^* \cap K' = \emptyset$. Therefore $C_N^* \in \mathcal{C}^*(M)$, and given that $C_N^* \cap C = \emptyset$, it holds that $|C_N^*| \geq d$. Since $|C_N| = |C| \leq 2d - 1$, it holds that $|C_N| \leq \min\{r(N), 2d - 1\}$. Now $|E(N)| < |E(M)|$, and hence there is a C_N-augmenting circuit D_N in N. Note that D_N is a circuit of M. We shall examine three cases.

Case 1: Suppose $D_N \cap C_N \subset D_0$.

We claim that $D_N \triangle D \in \mathcal{C}(M)$. For if not, then there exist disjoint circuits $D'_N, D''_N \subseteq D_N \triangle D$, where $D'_N, D''_N \in \mathcal{C}(M)$. We have $D'_N \cap D_1 \neq \emptyset$; otherwise, $D'_N \subset D_N \triangle C_N$, and given that $D'_N \neq D_N, C_N, D_N \triangle C_N$, this would contradict the fact that D_N is C_N-augmenting (because D_N must also be a C_N-minimal circuit). Thus $D'_N \cap D_1 \neq \emptyset$ and similarly, $D''_N \cap D_1 \neq \emptyset$. Thus we have $\emptyset \neq D'_N \cap D_1 = D'_N \cap K' \subset D_1$. Since D_0 and D_1 belong to different components of M', it follows that D'_N contains a circuit of M', not equal to D_1, which contains elements of D_1. Such a circuit must necessarily be contained in K'. However, such a circuit must also contain elements of $D'_N \backslash D_1 \subset E(N)$, which is impossible. We conclude that $D_N \triangle D \in \mathcal{C}(M)$.

We shall show that $D_N \triangle D$ is a C-minimal circuit. Suppose this is not the case. Then there exists $D'_N \subset (D_N \triangle D) \cup C$ where $D'_N \neq D \triangle D_N, C_N \triangle D_N, C$. It must hold that $D'_N \cap D_1 \neq \emptyset$, for otherwise, $D'_N \subset D_N \cup C_N$, and given that $D'_N \neq D_N, C_N \triangle D_N, C_N$, this would contradict the fact that D_N is a C_N-minimal circuit. For similar reasons, we also have $(D'_N \triangle D_N \triangle D) \cap D_1 \neq \emptyset$. Thus $\emptyset \neq D'_N \cap D_1 = D'_N \cap K' \subset D_1$. It is seen that $D'_N \backslash D_2$ contains a circuit of K' other than D_1. However, $D_N \cap K' \subset D_1$, and thus $D'_N \cap K'$ is independent in M'. This gives a contradiction. We conclude that $D_N \triangle D$ is a C-minimal circuit. Since $|C \triangle (D_N \triangle D)| = |C_N \triangle D_N| > |C_N| = |C|$, $D_N \triangle D$ is C-augmenting. This contradicts our assumption. Thus $C_N \cap D_N \not\subset D_0$.

Case 2: Suppose $C_N \cap D_N \subset D_2$.

Using similar arguments as in Case 1, one can show that D_N is a C-minimal circuit. We have that

$$|C \triangle D_N| = |C \triangle D_N| - |D_0| + |D_1| = |C_N \triangle D_N| > |C_N| = |C|.$$

In this case, D_N is a C-augmenting circuit, contradicting our assumption.

Case 3: Suppose $D_N \cap D_0 \neq \emptyset$ and $D_N \cap D_2 \neq \emptyset$.

Using similar arguments as in Case 1, one can show that $D_N \triangle D$ is a C-minimal circuit. We have $|C \triangle (D_N \triangle D)| = |C_N \triangle D_N| > |C_N| = |C|$. In this case, $D_N \triangle D$ is C-augmenting, contradicting our assumptions.

From Cases 1, 2, and 3 we conclude that C_N is a spanning circuit of N. □

5. Chord circuits

Let N and $C_N = D_0 \cup D_2$ be as in Section 4. By (4.5), C_N is a spanning circuit of N. Then each $e \in E(N)\backslash C_N$ is a **chord** of C_N; that is, $e \in \mathrm{cl}_N(C_N)$. Thus there are exactly two circuits in $C_N \cup \{e\}$, denoted C_e^1 and C_e^2, which contain e. We shall call C_e^1 and C_e^2 the **chord circuits** for e. We have $C_e^j = C_e^{3-j} \triangle C_N, j = 1, 2$. Let $\widetilde{C}_e^j = C_e^j \backslash \{e\},\ j = 1, 2$. We say that two chords e and f **cross** if $\widetilde{C}_e^1 \curlywedge \widetilde{C}_f^1$. A chord circuit C_e^i is said to be **minimal** if $\widetilde{C}_f^j \not\subset \widetilde{C}_e^i$, for all $f \in E(N)\backslash C_N$, and for all $j \in \{1, 2\}$. In this section, we shall assume that $a \in E(N)\backslash C_N$ is a chord of C_N such that C_a^1 is a minimal chord circuit. Furthermore, we may assume that $C_a^1 \subseteq D_0 \cup \{a\}$; otherwise, if no such chord a exists, we may just add an element a to N such that $D_0 \cup \{a\}$ is a circuit of N. We note that in such an instance $N \cup \{a\}$ is still regular since $N \cup D_1$ is a minor of M (and hence also regular). We remark also that $|E(N) \cup \{a\}| < |E(M)|$ since $|D_1| \geq 2$. Thus we may assume such a chord a exists. We let

$$A = \tilde{C}_a^1, \quad B = \tilde{C}_a^2, \quad X = \{x \in E(N)\backslash C_N \mid x \text{ and } a \text{ cross}\}.$$

For each $x \in X$, and $j \in \{1, 2\}$, we define the sets $A_x^j = \widetilde{C}_x^j \cap A$, $B_x^j = \widetilde{C}_x^j \cap B$. Let $\mathcal{A} = \{A_x^j \mid x \in X, j \in \{1, 2\}\}$.

Lemma 5.1. (i) *$\mathcal{A}$ is a non-crossing collection of subsets of A.*

(ii) *If $A_x^i \diamond A_y^j$, then $B_x^i \diamond B_y^j$. Furthermore, if $A_x^i = A_y^j$, then either $B_x^i \subset B_y^j$ or $B_y^j \subset B_x^i$.*

(iii) *If $A_x^i \parallel A_y^j$, then $B_x^i \parallel B_y^j$.*

Proof. (i) Let $x, y \in X$. It suffices to show that $A_x^1 \not\curlywedge A_y^1$. Suppose to the contrary that $A_x^1 \curlywedge A_y^1$. Then there exist elements a_1, a_2, a_3, a_4 where $a_1 \in A_x \cap A_y$, $a_2 \in A_x \backslash A_y$, $a_3 \in A_y \backslash A_x$, and $a_4 \in A \backslash (A_x \cup A_y)$.

Case 1: Suppose $B_x^1 \not\diamond B_y^1$.

There exist $b_1 \in B_x^1 \backslash B_y^1$ and $b_2 \in B_y^1 \backslash B_x^1$. We have that $a_1 \in C_a^1 \cap C_x^1 \cap C_y^1$, and $a_4 \in C_a^1 \backslash (C_x^1 \cup C_y^1)$. We also have that $b_1 \in C_x^1 \backslash (C_a^1 \cup C_y^1)$ and $b_2 \in C_y^1 \backslash (C_a^1 \cup C_x^1)$. Consequently, $(a_4, b_1, b_2) \underset{C_N, a_1}{\rightarrow} (C_a^1, C_x^1, C_y^1)$. By Lemma 2.2, N has an F_7-minor, a contradiction.

Case 2: Suppose $B_x^1 \diamond B_y^1$.

Without loss of generality, we may assume $B_x^1 \subseteq B_y^1$. Let $b_1 \in B_x^1$, and $b_2 \in B_y^2 = B \backslash B_y^1$. Then $a_2 \in C_x^1 \backslash (C_a^2 \cup C_y^1)$ and $a_3 \in C_y^1 \backslash (C_a^2 \cup C_x^1)$. Furthermore, $b_1 \in C_a^2 \cap C_x^1 \cap C_y^1$ and $b_2 \in C_a^2 \backslash (C_x^1 \cup C_y^1)$. Consequently, $(b_2, a_2, a_3) \underset{C_N, b_1}{\rightarrow} (C_a^2, C_x^1, C_y^1)$. Again this gives a contradiction.

From Cases 1 and 2 we conclude that $A_x^1 \not\curlywedge A_y^1$. $\square$

From (i) it follows that for all $A_x^i, A_y^j \in \mathcal{A}$, either $A_x^i \diamond A_y^j$ or $A_x^i \parallel A_y^j$.

Proof. (ii) By contradiction. Suppose $A_x^i \subseteq A_y^j$ and $B_x^i \not\diamond B_y^j$. Then $\exists b_1 \in B_x^i \backslash B_y^j$ and $\exists b_2 \in B_y^j \backslash B_x^i$. Let $a_1 \in A_x^i$ and $a_2 \in A \backslash A_y^j$; a_2 exists since y crosses a and hence $A \curlywedge A_y^j$. Now it is seen that $a_1 \in C_a^1 \cap C_x^i \cap C_y^j$, $a_2 \in C_a^1 \backslash (C_x^i \cup C_y^j)$, $b_1 \in C_x^i \backslash (C_a^1 \cup C_y^1)$, and $b_2 \in C_y^j \backslash (C_a^1 \cup C_x^i)$. Consequently, $(a_2, b_1, b_2) \underset{C_N, a_1}{\rightarrow} (C_a^1, C_x^i, C_y^j)$. This gives a contradiction. A similar contradiction is reached if we assume $A_y^j \subseteq A_x^i$. We conclude that if $A_x^i \diamond A_y^j$, then $B_x^i \diamond B_y^j$. If $A_x^i = A_y^j$, and $B_x^i = B_y^j$, then $C_x^i \triangle C_y^j = \{x, y\}$. In this case, $\{x, y\}$ is a digon, contradicting the assumption that N is simple. Thus if $A_x^i = A_y^j$, then either $B_x^i \subset B_y^j$ or $B_y^j \subset B_x^i$. $\square$

Proof. (iii) By contradiction. Suppose $A_x^i \parallel A_y^j$ and $B_x^i \nparallel B_y^j$. We have $A_x^i \not\diamond A_y^j$. Let $a_1 \in A_x^i \backslash A_y^j$ and $a_2 \in A_y^j \backslash A_x^i$. We have $B_x^i \cap B_y^j \neq \emptyset$ and $B \backslash (B_x^i \cup B_y^j) \neq \emptyset$. Let $b_1 \in B_x^i \cap B_y^j$ and $b_2 \in B \backslash (B_x^i \cup B_y^j)$. Then it is seen that $a_1 \in C_x^i \backslash (C_a^2 \cup C_y^j)$, $a_2 \in C_y^j \backslash (C_a^2 \cup C_x^i)$, $b_1 \in C_a^2 \cap C_x^i \cap C_y^j$, and $b_2 \in C_a^2 \backslash (C_x^i \cup C_y^j)$. Consequently, $(a_1, a_2, b_2) \underset{C_N, b_1}{\rightarrow} (C_x^i, C_y^j, C_a^2)$. This gives a contradiction. $\square$

For each $A_x^i \in \mathcal{A}$ let $\alpha(A_x^i)$ denote the maximum integer k such that there is a chain of sets $A_x^i \subset A_{x_1}^{i_1} \subset \cdots \subset A_{x_{k-1}}^{i_{k-1}} \subset A$. Let $n = \max_{A_x^i \in \mathcal{A}}\{\alpha(A_x^i)\}$. Let $x_0 \in X$ where $n = \max\{\alpha(A_{x_0}^1), \alpha(A_{x_0}^2)\}$. Without loss of generality, we may assume $n = \alpha(A_{x_0}^1)$. Let $e_0 \in A_{x_0}^1$. By reindexing if necessary, we may assume that $e_0 \in C_x^1$ for all chords x of C_N. Then $e_0 \in A_x^1$ for all $x \in X$. To simplify our notation, for each $x \in X$ we let

$$A_x = A_x^1, \qquad B_x = B_x^1, \qquad C_x = C_x^1, \qquad \widehat{A}_x = A_x^2, \qquad \widehat{B}_x = B_x^2, \qquad \widehat{C}_x = C_x^2, \qquad \widetilde{C}_x = \widetilde{C}_x^1.$$

For each $x \in X$, we define the sets

$$[x] = \{x' \in X \mid A_{x'} = A_x\}, \qquad \mathcal{B}_x = \{B_{x'} \mid A_{x'} = A_x\},$$
$$\mathcal{C}_x = \{C_{x'} \mid A_{x'} = A_x\}, \qquad \widetilde{\mathcal{C}}_x = \{\widetilde{C}_{x'} \mid A_{x'} = A_x\}.$$

We say that a finite collection of sets $\mathcal{S}$ is **strictly nested** if for some ordering of the sets $S_1, S_2, \ldots, S_n$ it holds $S_1 \subset S_2 \subset \cdots \subset S_n$. That is, the sets of $\mathcal{S}$ are totally ordered by inclusion.

Lemma 5.2. *For each $x \in X$, both $\mathcal{B}_x$ and $\widetilde{\mathcal{C}}_x$ are strictly nested collections of sets.*

Proof. Suppose $x' \in [x]$. Then $A_x' = A_x$, and hence Lemma 5.1(ii) implies that either $B_{x'} \subset B_x$ or $B_x \subset B_{x'}$. Thus the collection $\mathcal{B}_x$ is totally ordered under inclusion. From this, we conclude that $\mathcal{B}_x$ and hence also $\widetilde{\mathcal{C}}_x$ is a strictly nested collection of sets. $\square$

For each pair of elements $e, f \in D_0$, we define $C_{e,f}^* = E(N)\backslash(\mathrm{cl}(C_N\backslash\{e, f\}))$. Given that C_N is a spanning circuit of N, we have $r(C_N\backslash\{e, f\}) = r(N) - 1$. Thus the set $C_{e,f}^*$ is seen to be a cocircuit of N and hence it is a cocircuit of N/D_2 as well (since $C_{e,f}^* \cap D_2 = \emptyset$). To show that $C_{e,f}^*$ is also a cocircuit of M, recall that $N = M\backslash K'$ where K' is the component of M/D_2 containing D_1. Thus $N/D_2 = M\backslash K'/D_2 = M/D_2\backslash K'$. Since $C_{e,f}^* \not\subseteq K'$, it follows that $C_{e,f}^*$ is a cocircuit of M/D_2. Thus it must also be a cocircuit of M. We let $\widetilde{C}_{e,f}^* = C_{e,f}^*\backslash\{e, f\}$. It should be noted that $\widetilde{C}_{e_0,f}^* \subseteq X$ for all $f \in A\backslash\{e_0\}$. To see this, let $f \in A\backslash\{e_0\}$ and $x \in \widetilde{C}_{e_0,f}^*$. Then $e_0 \in C_x$ and $f \in \widehat{C}_x$. Since C_a is a minimum chord circuit, it follows that B_x and $\widehat{B}_x$ are nonempty. Thus all of the sets $A_x, \widehat{A}_x, B_x$, and $\widehat{B}_x$ are nonempty and consequently x crosses a. Now $x \in X$ and hence $X \subseteq \widetilde{C}_{e_0,f}^*$. The next lemma will be used later to find a C-augmenting circuit.

Lemma 5.3. *The circuits C_x and $\widehat{C}_x$ are C-minimal circuits for all $x \in X$ where $C_x \cap C \neq \emptyset$ and $\widehat{C}_x \cap C \neq \emptyset$. Furthermore, $C_x \triangle C_y$ is a C-minimal circuit for all $x, y \in X$ where $A_x \subseteq A_y$ and $(C_x \triangle C_y) \cap C \neq \emptyset$.*

Proof. We shall only prove the second assertion, as the first assertion has an easier proof which uses similar arguments. Let $x, y \in X$ where $A_x \subseteq A_y$ and $(C_x \triangle C_y) \cap C \neq \emptyset$. Let $\Omega = C_x \triangle C_y$ and let Ω' be a circuit where $\Omega' \subset \Omega \cup C$. If $\Omega' \cap (\Omega\backslash C) = \emptyset$, then clearly $\Omega' = C$. Suppose $\Omega\backslash C \subseteq \Omega'$. Then $\Omega \triangle \Omega' \subseteq C$ and hence $\Omega \triangle \Omega' = \emptyset$ or C. In this case, $\Omega' = \Omega$ or $\Omega' = \Omega \triangle C$. For the remainder, we may assume that $\Omega' \cap (\Omega\backslash C) \neq \emptyset$ and $\Omega\backslash C \not\subseteq \Omega'$. Suppose $x \in \Omega'$ and $y \notin \Omega'$. Let $f \in \widehat{A}_y$. Then $x, y \in C_{e_0,f}^*$ since $A_x \subseteq A_y$. Now $C_{e_0,f}^* \cap \Omega' = \{x\}$, which is impossible since the nonempty intersection of circuit and cocircuit must contain at least two elements. A similar argument applies when $x \notin \Omega'$ and $y \in \Omega'$. If $x, y \notin \Omega'$, then $\Omega' \subseteq (D_0\backslash\{e_0\}) \cup C$ and $\Omega' \cap (D_0\backslash\{e_0\}) \neq \emptyset$. Then $\Omega' \neq C, D, C \triangle D$, contradicting the fact that D is C-minimal. If $x, y \in \Omega'$, then $\Omega \triangle \Omega'$ contains a circuit Ω'' where $\Omega'' \cap (D_0\backslash\{e_0\}) \neq \emptyset$. Then $\Omega'' \neq C, D, C \triangle D$, and again, this contradicts the fact that D is C-minimal. From the above, we conclude that $\Omega' = \Omega, C, C \triangle \Omega$ and hence Ω' is C-minimal. $\square$

The following theorem will be instrumental in the proof of the main theorem.

Theorem 5.4. *There exists $f_0 \in A\backslash\{e_0\}$ such that the collection $\{\widetilde{C}_x \mid x \in \widetilde{C}_{e_0,f_0}^*\}$ is strictly nested.*

Proof. In view of Lemma 5.2, we may assume that $B_x \subset B_{x_0}$ for all $x \in [x_0]\backslash\{x_0\}$. Suppose $x \in X\backslash\{x_0\}$. By the choice of x_0 we have $A_x \not\subset A_{x_0}$; otherwise, $\alpha(A_x) > \alpha(A_{x_0})$. Similarly, $\widehat{A}_x \not\subset A_{x_0}$. Furthermore, $A_{x_0} \not\parallel A_x$ by Lemma 5.1(i). By our assumptions, we also have $e_0 \in A_{x_0} \cap A_x$. The proof of the theorem is divided into a number of subproofs. $\square$

(5.5). *$A_{x_0} \subseteq A_x$ and $B_{x_0} \diamond B_x$ for all $x \in X$.*

Proof. By contradiction. Let $x \in X$. Suppose $A_{x_0} \nsubseteq A_x$. Since $A_x \not\perp A_{x_0}$, it follows that $A_x \diamond A_{x_0}$ or $A_x \parallel A_{x_0}$. Since $A_{x_0} \nsubseteq A_x$ and $A_x \not\subset A_{x_0}$, it follows that $A_x \parallel A_{x_0}$. Given that $e_0 \in A_x \cap A_{x_0}$, it must hold that $A_{x_0} \cup A_x = A$. This implies that $\widehat{A}_x \subset A_{x_0}$, noting that $\widehat{A}_x \neq A_{x_0}$ since $e_0 \notin \widehat{A}_x$. This gives a contradiction. We conclude that $A_{x_0} \subseteq A_x$ and $A_{x_0} \diamond A_x$. Now Lemma 5.1 (ii) implies that $B_{x_0} \diamond B_x$. □

(5.6). *If $n = 1$, then $\{\widetilde{C}_x \mid x \in \widetilde{C}^*_{e_0,f}\}$ is a strictly nested collection for any $f \in \widehat{A}_{x_0}$.*

Proof. Suppose $n = 1$. Then by (5.5), we must have $X = [x_0]$. Furthermore, for $f \in \widehat{A}_{x_0}$ we have $\{\widetilde{C}_x \mid x \in \widetilde{C}^*_{e_0,f}\} = \widetilde{\mathcal{C}}_{x_0}$. Now Lemma 5.2 implies that $\{\widetilde{C}_x \mid x \in \widetilde{C}^*_{e_0,f}\}$ is strictly nested. □

If $|C| = r(M)$, then $|D_0| = 2$ by (4.3). In this case, $n = 1, X = [x_0]$, and the theorem is seen to hold (by (5.6)). Thus we may assume for the remainder of the proof that $|C| < r(M)$ and $n \geq 2$. Let $y_0 \in A$ where $\alpha(A_{y_0}) = n - 1$. By (5.5), we have $A_{x_0} \subset A_{y_0}$. Let $f_0 \in A_{y_0} \backslash A_{x_0}$.

(5.7). *Suppose that for all $x \in X$ where $\widehat{A}_x \subseteq A_{y_0} \backslash A_{x_0}$ it holds that $f_0 \notin \widehat{A}_x$. Then $\widetilde{C}^*_{e_0,f_0} = [x_0]$, and $\{\widetilde{C}_x \mid x \in \widetilde{C}^*_{e_0,f_0}\}$ is strictly nested.*

Proof. Assume that for all $x \in X$ where $\widehat{A}_x \subseteq A_{y_0} \backslash A_{x_0}$ it holds that $f_0 \notin \widehat{A}_x$. Since $e_0 \in A_x$ and $f_0 \in \widehat{A}_x = \widehat{A}_{x_0}$ for all $x \in [x_0]$, it is clear that $[x_0] \subseteq \widetilde{C}^*_{e_0,f_0}$. We shall show that $\widetilde{C}^*_{e_0,f_0} = [x_0]$. Let $x \in \widetilde{C}^*_{e_0,f_0}$. Since $e_0 \in A_x$, it must hold that $f_0 \in \widehat{A}_x$. Suppose $x \in X \backslash [x_0]$. By (5.5), $A_{x_0} \subset A_x$. This implies that $\alpha(A_x) \leq n - 1 = \alpha(A_{y_0})$. Thus $A_x \not\subset A_{y_0}$; otherwise, $A_{x_0} \subset A_x \subset A_{y_0}$, implying that $\alpha(A_x) > \alpha(A_{y_0})$. We also have that $A_{y_0} \nsubseteq A_x$ since $f_0 \in \widehat{A}_x \cap A_{y_0}$. Since $A_x \not\perp A_{y_0}$ (by Lemma 5.1(i)), we have that $A_x \diamond A_{y_0}$ or $A_x \parallel A_{y_0}$. Since $A_x \nsubseteq A_{y_0}$ and $A_{y_0} \nsubseteq A_x$, it follows that $A_x \not\diamond A_{y_0}$ and hence $A_x \parallel A_y$. Given that $e_0 \in A_x \cap A_{y_0}$, it follows that $A_x \cup A_{y_0} = A$. Consequently, $\widehat{A}_x \subseteq A_{y_0} \backslash A_{x_0}$. However, $f_0 \in \widehat{A}_x$, and this contradicts our assumptions. We conclude that $x \in [x_0]$ and hence $\widetilde{C}^*_{e_0,f_0} \subseteq [x_0]$. This means that $\widetilde{C}^*_{e_0,f_0} = [x_0]$. It now follows from Lemma 5.2 that $\{\widetilde{C}_x \mid x \in \widetilde{C}^*_{e_0,f_0}\}$ is a strictly nested. □

In light of (5.7), we may assume for the remainder that there exists $z_0 \in X$ such that $\widehat{A}_{z_0} \subseteq A_{y_0} \backslash A_{x_0}$ and $f_0 \in \widehat{A}_{z_0}$. By Lemma 5.2, we may also assume that $B_z \supset B_{z_0}$ for all $z \in [z_0] \backslash \{z_0\}$. We also observe that $\alpha(\widehat{A}_{z_0}) = n = \alpha(A_{x_0})$ since $\alpha(A_{y_0}) = n - 1$.

(5.8). *For all $x \in [x_0]$, $y \in [y_0]$, and $z \in X$ where $\widehat{A}_z \subseteq A_{y_0} \backslash A_{x_0}$ it holds that $B_x \diamond B_y$, $B_x \diamond B_z$, and $B_y \parallel B_z$.*

Proof. Let $x \in [x_0]$, $y \in [y_0]$, and $z \in X$ where $\widehat{A}_z \subseteq A_{y_0} \backslash A_{x_0}$. By (5.5) we have that $B_x \diamond B_y$ and $B_x \diamond B_z$. Also, it is seen that $A_y \cup A_z = A$. Thus we have $A_y \parallel A_z$. Now it follows by Lemma 5.1(iii) that $B_y \parallel B_z$. □

(5.9). *If $B_{x_0} \subseteq B_{y_0}$, then $\widetilde{C}^*_{e_0,f_0} = [x_0] \cup [z_0]$ and $\{\widetilde{C}_x \mid x \in \widetilde{C}^*_{e_0,f_0}\}$ is strictly nested.*

Proof. Assume that $B_{x_0} \subseteq B_{y_0}$. We first observe that $[x_0] \cup [z_0] \subseteq \widetilde{C}^*_{e_0,f_0}$. To see this, let $x \in [x_0] \cup [z_0]$. Then $e_0 \in A_x$ and $f_0 \in \widehat{A}_x$. This means that $x \notin \mathrm{cl}_N(C_N \backslash \{e_0, f_0\})$. Thus $x \in E(N) \backslash (\mathrm{cl}_N)((C_N \backslash \{e_0, f_0\}) \cup \{e_0, f_0\}) = \widetilde{C}^*_{e_0,f_0}$. We shall show that $\widetilde{C}^*_{e_0,f_0} = [x_0] \cup [z_0]$. Let $x \in \widetilde{C}^*_{e_0,f_0}$. Given that $e_0 \in A_x$, it must hold that $f_0 \in \widehat{A}_x$. Suppose that $x \notin [x_0] \cup [z_0]$. By (5.5), $A_{x_0} \subset A_x$. As in the proof of (5.7), we deduce that $\widehat{A}_x \subseteq A_{y_0} \backslash A_{x_0}$. Then $\alpha(\widehat{A}_x) = n$. We have $f_0 \in \widehat{A}_x \cap \widehat{A}_{z_0}$ and $\widehat{A}_x \cup \widehat{A}_{z_0} \subset A$ (since $e_0 \in A \backslash (\widehat{A}_x \cup \widehat{A}_{z_0})$). Thus $A_x \nparallel A_{z_0}$. Since $\widehat{A}_x \not\perp \widehat{A}_{z_0}$, it follows that $\widehat{A}_x \diamond \widehat{A}_{z_0}$. We note that $\widehat{A}_x \neq \widehat{A}_{z_0}$ since $x \notin [z_0]$. Also, $\widehat{A}_x \not\subset \widehat{A}_{z_0}$ and $\widehat{A}_{z_0} \not\subset \widehat{A}_x$ since $\alpha(\widehat{A}_{z_0}) = \alpha(\widehat{A}_x) = n$. This yields a contradiction. We conclude that $x \in [x_0] \cup [z_0]$, and hence $\widetilde{C}^*_{e_0,f_0} = [x_0] \cup [z_0]$. By (5.8), we have $B_{x_0} \diamond B_{z_0}$ and $B_{y_0} \parallel B_{z_0}$. Suppose that $B_{z_0} \subseteq B_{x_0}$. Then $B_{z_0} \subseteq B_{x_0} \subseteq B_{y_0}$, implying that $B_{y_0} \diamond B_{z_0}$, which yields a contradiction since $B_{y_0} \parallel B_{z_0}$. Thus $B_{x_0} \subset B_{z_0}$. By our choice of x_0 and z_0, $B_x \subset B_{x_0}$ for all $x \in [x_0] \backslash \{x_0\}$ and $B_{z_0} \subset B_z$ for all $z \in [z_0] \backslash \{z_0\}$. It follows that $B_x \subset B_z$ for all $x \in [x_0]$ and $z \in [z_0]$. Since both $\widetilde{\mathcal{C}}_{x_0}$ and $\widetilde{\mathcal{C}}_{z_0}$ are both strictly nested collections (by Lemma 5.2), it follows that $\widetilde{\mathcal{C}}_{x_0} \cup \widetilde{\mathcal{C}}_{z_0}$ is a strictly nested collection. We conclude that $\{\widetilde{C}_x \mid x \in \widetilde{C}^*_{e_0,f_0}\}$ is a strictly nested. $\square$

(5.10). *Let $x \in [x_0]$, $y \in [y_0]$, and $z \in X$ where $\widehat{A}_z \subseteq A_{y_0} \backslash A_{x_0}$. If $B_y \subset B_x$, then $z = z_0$, $[z_0] = \{z_0\}$ and $B_x = B_y \overset{\circ}{\cup} B_{z_0}$.*

Proof. Let $x \in [x_0]$, $y \in [y_0]$, and $z \in X$ where $\widehat{A}_z \subseteq A_{y_0} \backslash A_{x_0}$. Assume that $B_y \subset B_x$. We observe that $\alpha(\widehat{A}_z) = n$, since $\widehat{A}_z \subset A_{y_0}$. We have that $B_x \diamond B_z$ and $B_y \parallel B_z$ by (5.8). If $B_x \subseteq B_z$, then $B_y \subset B_x \subseteq B_z$ implying that $B_y \diamond B_z$, contradicting the fact that $B_y \parallel B_z$. Thus it holds that $B_x \not\subseteq B_z$ and hence $B_z \subset B_x$. This means that $B_y \cup B_z \subseteq B_x \subset B$. Since $B_y \parallel B_z$, it follows that $B_y \cap B_z = \emptyset$.

Suppose $B_x \backslash (B_y \cup B_z) \neq \emptyset$. Let $a_1 \in \widehat{A}_y$, $b_1 \in B_x \backslash (B_y \cup B_z)$, and $f \in \widehat{A}_z$. We have that $e_0 \in C_x \cap C_y \cap C_z$ and $f \in C_y \backslash (C_x \cup C_z)$. Since $a_1 \in \widehat{A}_y$ and $\widehat{A}_y \subset A_z$, it follows that $a_1 \in C_z \backslash (C_x \cup C_y)$. Furthermore, since $b_1 \in B_x \backslash (B_y \cup B_z)$, it follows that $b_1 \in C_x \backslash (C_y \cup C_z)$. Consequently, $(f, a_1, b_1) \underset{C_N, e_0}{\rightarrow} (C_y, C_z, C_x)$. This gives a contradiction. From this we conclude that $B_x \backslash (B_y \cup B_z) = \emptyset$ and hence $B_x = B_y \overset{\circ}{\cup} B_z$.

It is seen that the above holds for z_0; that is, $B_x = B_y \overset{\circ}{\cup} B_{z_0}$. This in turn means that $B_z = B_{z_0}$ and hence $B_z \diamond B_{z_0}$. It now follows from Lemma 5.1(ii) and (iii) that $A_z \diamond A_{z_0}$, and hence $\widehat{A}_z \diamond \widehat{A}_{z_0}$ as well. However, since $\alpha(\widehat{A}_z) = \alpha(\widehat{A}_{z_0}) = n$, it follows that $\widehat{A}_z \not\subset \widehat{A}_{z_0}$ and $\widehat{A}_{z_0} \not\subset \widehat{A}_z$. Consequently, it must hold that $\widehat{A}_z = \widehat{A}_{z_0}$ and $\widetilde{C}_z = \widetilde{C}_{z_0}$. Now if $z \neq z_0$, then $C_z \triangle C_{z_0} = \{z, z_0\}$ is a digon in N, contradicting the assumption that N is simple. Thus it must hold that $z = z_0$. Since this holds for any $z \in [z_0]$, we have $[z_0] = \{z_0\}$. $\square$

(5.11). *If $B_{y_0} \subset B_{x_0}$, then $[x_0] = \{x_0\}$, $[y_0] = \{y_0\}$, and $[z_0] = \{z_0\}$.*

Proof. Assume that $B_{y_0} \subset B_{x_0}$. By (5.10), we have that $B_{x_0} = B_{y_0} \overset{\circ}{\cup} B_{z_0}$ and $[z_0] = \{z_0\}$. Let $x \in [x_0]$. Then $B_x \diamond B_{y_0}$ by (5.8). If $B_x \subseteq B_{y_0}$, then $B_x \cap B_{z_0} \subseteq B_{y_0} \cap B_{z_0} = \emptyset$ implying that $B_x \parallel B_{z_0}$, a contradiction. Thus $B_x \not\subseteq B_{y_0}$, and hence $B_{y_0} \subset B_x$. Now (5.10) implies that $B_{y_0} \overset{\circ}{\cup} B_{z_0} = B_x$. Moreover, since x was arbitrarily chosen, this holds for all $x \in [x_0]$. It follows that $B_x = B_{x_0}$ and $\widetilde{C}_x = \widetilde{C}_{x_0}$ for all $x \in [x_0]$. If there exists $x \in [x_0] \backslash \{x_0\}$, then $C_x \triangle C_{x_0} = \{x, x_0\}$ is a digon of N. This contradicts the assumption that N is simple. We conclude that $[x_0] = \{x_0\}$. To show $[y_0] = \{y_0\}$, suppose to the contrary that there exists $y \in [y_0] \backslash \{y_0\}$. By (5.8), we have $B_{x_0} \diamond B_y$ and $B_y \parallel B_{z_0}$. If $B_{x_0} \subseteq B_y$, then $B_{z_0} \subset B_y$, since $B_{z_0} \subset B_{x_0}$. This contradicts the fact that $B_y \parallel B_{z_0}$. Thus $B_{x_0} \not\subseteq B_y$ and hence $B_y \subset B_{x_0}$. Now (5.10) implies that $B_{x_0} = B_y \overset{\circ}{\cup} B_{z_0}$. In particular, $B_y = B_{y_0} = B_{x_0} \backslash B_{z_0}$. Thus $\widetilde{C}_y = \widetilde{C}_{y_0}$. Now $C_y \triangle C_{y_0} = \{y, y_0\}$ is a digon, contradicting the assumption that N is simple. We conclude that $[y_0] = \{y_0\}$. $\square$

(5.12). *Assuming $B_{y_0} \subset B_{x_0}$, if $x \in X \backslash \{x_0, y_0, z_0\}$, then $A_{y_0} \subset A_x$.*

Proof. Assume that $B_{y_0} \subset B_{x_0}$ and let $x \in X \backslash \{x_0, y_0, z_0\}$. Since $x \neq z_0$, (5.10) implies that $\widehat{A}_x \not\subseteq A_{y_0} \backslash A_{x_0}$. We have $A_x \not\subset A_{y_0}$; otherwise by (5.5) we have $A_{x_0} \subset A_x \subset A_{y_0}$, implying that $\alpha(A_{x_0}) \geq \alpha(A_{y_0}) + 2 = n + 1$. Thus $A_x \backslash A_{y_0} \neq \emptyset$. Now if $A_{y_0} \backslash A_x \neq \emptyset$, then $A_x \not\diamond A_{y_0}$ and consequently, $A_x \parallel A_{y_0}$. This would mean that $A_{y_0} \cup A_x = A$ since $e_0 \in A_{y_0} \cap A_x$. It would follow that $\widehat{A}_x \subseteq A_{y_0}$ and hence $\widehat{A}_x \subset A_{y_0} \backslash A_{x_0}$ since $A_{x_0} \subset A_x$. This gives a contradiction. Thus $A_{y_0} \backslash A_x = \emptyset$ and hence $A_{y_0} \subset A_x$. $\square$

(5.13). *If $B_{y_0} \subset B_{x_0}$, then $X = \{x_0, y_0, z_0\}$.*

Proof. Assume that $B_{y_0} \subset B_{x_0}$ and let $x \in X \backslash \{x_0, y_0, z_0\}$. By (5.10), $B_{x_0} = B_{y_0} \overset{\circ}{\cup} B_{z_0}$. By (5.12), $A_{y_0} \subset A_x$. Thus $\widehat{A}_x \subset \widehat{A}_{y_0} \subset A_{z_0}$ by the choice of z_0. This means that $A_{z_0} \backslash A_x \neq \emptyset$ and $A_x \backslash A_{z_0} \neq \emptyset$. Then $A_x \not\diamond A_{z_0}$ and hence $A_x \parallel A_{z_0}$. It follows by Lemma 5.1(iii) that $B_x \parallel B_{z_0}$. Since $A_{x_0} \subset A_{y_0} \subset A_x$, we have that $B_x \diamond B_{x_0}$ (by Lemma 5.1(ii)). If $B_{x_0} \subseteq B_x$, then $B_{y_0} \subset B_{x_0} \subseteq B_x$. Hence, $B_{y_0} \overset{\circ}{\cup} B_{z_0} = B_{x_0} \subseteq B_x$. This implies that $B_{z_0} \subset B_x$, contradicting the fact that $B_x \parallel B_{z_0}$. Thus $B_{x_0} \not\subseteq B_x$ and hence $B_x \subset B_{x_0}$ and $B_x \cap B_{z_0} = \emptyset$. Let $a_1 \in A_x \backslash A_{y_0}$, $b_0 \in \widehat{B}_{x_0}$, $b_1 \in B_{z_0}$, and $b_2 \in B_x$. Then it is seen that $b_0 \in \widehat{C}_{x_0} \cap \widehat{C}_x \cap \widehat{C}_{z_0}$ (since $B_x \subset B_{x_0}$ and $B_{z_0} \subset B_{x_0}$) and $a_1 \in \widehat{C}_{x_0} \backslash (\widehat{C}_x \cup \widehat{C}_{z_0})$ (since $A_x \backslash A_{y_0} \subset A_x$ and $A_x \backslash A_{y_0} \subset A_{z_0}$). Furthermore, $b_1 \in \widehat{C}_x \backslash (\widehat{C}_{x_0} \cup \widehat{C}_{z_0})$ (since $B_{z_0} \subseteq B_{x_0} \cap \widehat{B}_x$) and $b_2 \in \widehat{C}_{z_0} \backslash (\widehat{C}_{x_0} \cup \widehat{C}_x)$. Thus we have $(a_1, b_1, b_2) \underset{C_N, b_0}{\rightarrow} (\widehat{C}_{x_0}, \widehat{C}_x, \widehat{C}_{z_0})$. This gives a contradiction. We conclude that no such x exists and thus $X = \{x_0, y_0, z_0\}$. $\square$

(5.14). $B_{y_0} \not\subset B_{x_0}$.

Proof. By contradiction. Assume that $B_{y_0} \subset B_{x_0}$. We shall show that M contains a C-augmenting circuit. By (5.10), we have $B_{x_0} = B_{y_0} \overset{\circ}{\cup} B_{z_0}$. We also have by (5.13) that $X = \{x_0, y_0, z_0\}$. Let $g_0 \in \widehat{A}_{y_0}$. It is straightforward to show that $C^*_{e_0,f_0} = \{e_0, f_0, x_0, z_0\}$. Furthermore, since $C^*_{e_0,f_0}$ is a cocircuit of M which is disjoint from C, we have that $d \le |C^*_{e_0,f_0}| = 4$. Thus $d \le 4$. Since $|C| \le 2d - 1$ (by assumption), we have that $|C| \le 7$. Given that $|C| = |D_0| + |D_2| \ge 2|D_0|$, it follows that $|D_0| \le 3$. Now $3 \le |A| \le |D_0| \le 3$, and hence $|A| = |D_0| = 3$. This means that $D_0 = A = \{e_0, f_0, g_0\}$ and $B = D_2$. Since $|C| = |D_0| + |D_2|$ and $|C| \le 7$, it follows that $|B| = |D_2| \le 4$. We observe that since B is the disjoint union of B_{x_0} and $\widehat{B}_{x_0}$, it is also the disjoint union of the nonempty sets $\widehat{B}_{x_0}$, B_{y_0}, and B_{z_0}. Since $|B| = |D_2| \le 4$, it follows that $|\widehat{B}_{x_0}| \le 2$, $|B_{y_0}| \le 2$, and $|B_{z_0}| \le 2$. Let $\Omega = C_{x_0} \triangle C_{y_0}$. Then it seen that $\Omega = \{x_0, y_0, f_0\} \cup B_{z_0}$, and furthermore Ω is a C-minimal by Lemma 5.3. Now $\Omega \triangle C$ is a circuit where $|\Omega \triangle C| = 3 + |C| - |B_{z_0}| \ge |C| + 1$. Thus Ω is seen to be a C-augmenting circuit, which yields a contradiction. We conclude that $B_{y_0} \not\subset B_{x_0}$.

We can now complete the proof of the theorem. By (5.14), $B_{y_0} \not\subset B_{x_0}$ and thus it follows that $B_{x_0} \subseteq B_{y_0}$. Now (5.9) implies that $\{\widetilde{C}_x \mid x \in \widetilde{C}^*_{e_0,f_0}\}$ is strictly nested. $\square$

6. Proof of the main theorem

In this section, we shall prove Theorem 3.2. Theorem 5.4 implies that there exists $f_0 \in D_0 \backslash \{e_0\}$ such that $\{\widetilde{C}_x \mid x \in \widetilde{C}^*_{e_0,f_0}\}$ is strictly nested. For convenience, let $\{C_x \mid x \in \widetilde{C}^*_{e_0,f_0}\} = \{C_i | i = 1, 2, \ldots, k\}$ where $\widetilde{C}_1 \subset \widetilde{C}_2 \subset \cdots \subset \widetilde{C}_k$. We shall define circuits $\Delta_1, \Delta_2, \ldots, \Delta_{l+1}$ in the following way:

$$\Delta_1 = C_1, \qquad \Delta_i = C_i \triangle C_{i-1}, \quad i = 2, \ldots, k \quad \text{and} \quad \Delta_{k+1} = \widehat{C}_k.$$

The sets $\Delta_i \cap (D_0 \cup D_2)$, $i = 1, \ldots, k+1$ form a partition of $D_0 \cup D_2$. We note that $\Delta_i \cap ((D_0 \cup D_2) \backslash \{e_0, f_0\}) \ne \emptyset$, $i = 1, \ldots, k+1$. Let $L = \{i \mid \Delta_i \cap D_2 \ne \emptyset\}$. Let $l = |L|$ and assume $L = \{i_1, \ldots, i_l\}$ where $i_1 < i_2 < \cdots < i_l$. We observe that if $i \in \{1, \ldots, k+1\} \backslash L$, then $\Delta_i \cap D_2 = \emptyset$ and hence $\Delta_i \cap (D_0 \backslash \{e_0, f_0\}) \ne \emptyset$. Since the sets $\Delta_i \cap (D_0 \backslash \{e_0, f_0\})$, $i \in \{1, \ldots, k+1\} \backslash L$ are disjoint, it follows that

$$k + 1 - l = |\{1, \ldots, k+1\} \backslash L| \le |D_0 \backslash \{e_0, f_0\}| = |D_0| - 2.$$

Thus

$$l \ge k + 3 - |D_0|. \tag{1}$$

Recall that $2d > |C| = |C_N| = |D_0| + |D_2|$ and $|C^*| \ge d$ for all cocircuits C^* of M where $C^* \cap C = \emptyset$. Since $C^*_{e_0,f_0} \cap C = \emptyset$, we have that $k + 2 = |C^*_{e_0,f_0}| \ge d$. Thus $d \le k + 2$. By (1), we obtain that

$$l \ge d + 1 - |D_0|. \tag{2}$$

Let $|D_2| = |D_0| + \delta$. We have $\delta \ge 0$ since $|D_2| \ge |D_0|$. Thus

$$2d > |C_N| = |D_0| + |D_2| = 2|D_0| + \delta$$
$$d > |D_0| + \frac{\delta}{2}$$
$$d - |D_0| > \frac{\delta}{2}.$$

This together with (2) implies that

$$l > \frac{\delta}{2} + 1. \tag{3}$$

We shall define circuits $\Delta'_1, \Delta'_2, \ldots, \Delta'_l$ in the following way:

$$\Delta'_1 = C_{i_1}, \qquad \Delta'_j = C_{i_j} \triangle C_{i_{j-1}}, \quad j = 2, \ldots, l-1, \qquad \Delta'_l = \begin{cases} \widehat{C}_{i_{l-1}}, & \text{if } i_l \le k; \\ \widehat{C}_k, & \text{if } i_l = k+1. \end{cases}$$

It is seen that $\Delta'_i \cap D_2 \neq \emptyset$ for all $i = 1, 2, \ldots, l$ and Δ'_i, $i = 1, \ldots, l$ are C-minimal circuits by Lemma 5.3. Since we are assuming that there are no C-augmenting circuits, we have that

$$|\Delta'_i \triangle C| \le |C|, \quad i = 1, \ldots, l. \tag{4}$$

We also observe that the sets $\Delta'_i \cap C_N$, $i = 1, \ldots, l$ form a partition of $C_N = D_0 \cup D_2$. In particular, this means that the sets $\Delta'_i \cap D_0$, $i = 1, \ldots, l$ partition D_0, and the sets $\Delta'_i \cap D_2$, $i = 1, \ldots, l$ partition D_2. Thus $\sum_{i=1}^{l} |\Delta'_i \cap D_0| = |D_0|$, and $\sum_{i=1}^{l} |\Delta'_i \cap D_2| = |D_2|$. We have that

$$|\Delta'_i \triangle C| = \begin{cases} |\Delta'_i \cap D_0| + |C| - |\Delta'_i \cap D_2| + 1, & \text{if } i \in \{1, l\}; \\ |\Delta'_i \cap D_0| + |C| - |\Delta'_i \cap D_2| + 2, & \text{if } i \in \{2, \ldots, l-1\}. \end{cases}$$

The above together with the inequalities in (4) imply

$$\begin{aligned} &|\Delta'_1 \cap D_0| + 1 \le |\Delta'_1 \cap D_2| \\ &|\Delta'_j \cap D_0| + 2 \le |\Delta'_j \cap D_2|, \quad j = 2, \ldots, l-1 \\ &|\Delta'_{l+1} \cap D_0| + 1 \le |\Delta'_l \cap D_2|. \end{aligned}$$

Summing the above inequalities, we obtain

$$\begin{aligned} &2(l-1) + \sum_{i=1}^{l} |\Delta'_i \cap D_0| \le \sum_{i=1}^{l} |\Delta'_j \cap D_2| \\ &2(l-1) + |D_0| \le |D_2| \\ &l - 1 \le \frac{|D_2| - |D_0|}{2} = \frac{\delta}{2}. \\ &l \le \frac{\delta}{2} + 1. \end{aligned}$$

The above inequality contradicts (3). We conclude that no such counterexample M can exist, and this completes the proof of Theorem 3.2.

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